Advanced Mathematical Models & Applications
Vol.4, No.1, 2019, pp.51-63

FLUID FLOW IN CURVILINEAR FLEXIBLE PIPES WITH LAYERED ELASTIC WALLS

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Abstract. A two-dimensional model describing the elastic properties of the wall of a curved flexible pipe is presented. This model takes the layered nature of the flexible pipe, the interaction of the wall with the surrounding material and the fluid flow into account and is built by means of a dimension reduction procedure. The considered object can be served as a model for an implant of blood vessel purpose-built from artificial materials for short-term using. In comparison with Ghosh et al. (2018) we consider the most general case of canonical shapes of pipes with a curved axis, variable radius, and equally spaced layers.

Keywords: fluid flow, curvilinear flexible pipe, asymptotic analysis, dimension reduction procedure.

AMS Subject Classification: 34C99, 35B40, 92C50.

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Received: 19 February 2019; Revised: 17 March 2019; Accepted: 27 March 2019; Published: 24 April 2019.

1 Introduction. Formulation of the problem

We consider a segment of a flexible pipe and denote the hollow interior by Ω. Let Γ denote the part of the flexible pipe wall surrounding the region Ω. Γ has a layered structure with the layers separated by surfaces $\Gamma_{\rm in}, \Gamma_1, \dots, \Gamma_m, \Gamma_{\rm out}$ where $\Gamma_{\rm in}$ denotes the interior boundary of Γ and $\Gamma_{\rm out}$ denotes the exterior boundary of Γ. In our considerations these layers are typically made of anisotropic elastic material. We assume that a central line is given and that it has a general geometry allowing curvature and torsion. We also assume that along the chosen central line, the pipe has a circular cross-section with slightly varying radius.

For some time interval $[0,T]$, let the velocity field of the channel inside the pipe be given by $\mathbf{v} : \Omega \times [0,T] \to \mathbb{R}^3$. The displacement field in the pipe shell is denoted by $\mathbf{u} : \Gamma \times [0,T] \to \mathbb{R}^3$, $\mathbf{u} = (u_1, u_2, u_3)$. Let $p : \Omega \times [0,T] \to \mathbb{R}$ denote the pressure associated with the fluid (blood) in the pipe.

The relation between the elastic stress tensor, denoted by $\sigma = \{\sigma_{ij}\}_{i,j=1}^3$, and the elastic strain tensor, denoted by $\varepsilon = \{\varepsilon_{ij}\}_{i,j=1}^3$, in the vessel wall is given by the Hooke's law

$$\sigma_{ij} = \sum_{k,l=1}^{3} A_{ij}^{kl} \varepsilon_{kl} \tag{1}$$

with A_{ij}^{kl} being the stiffness tensor having the symmetries $A_{ij}^{kl} = A_{ji}^{kl} = A_{ji}^{lk}$. We assume that A_{ij}^{kl} are constant across each layer, which can be different for different layers. The strain tensor and the displacement vector have the relation

$$\varepsilon_{kl} = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right). \tag{2}$$

From the equilibrium conditions in the pipe shell, we get the equation

$$\nabla \cdot \sigma = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} \text{ in } \Gamma, \tag{3}$$

where ρ is the pipe shell mass density which is assumed to be piecewise continuous.

On the inner boundary Γ_{in}, with h being a small parameter defined in the next section, we have

$$\sigma \mathbf{n} = h \rho_b \mathbf{F} \text{ and } \partial_t \mathbf{u} = \mathbf{v}, \tag{4}$$

where ρ_b is the blood density and $\mathbf{F}$ is the hydrodynamic force in the blood, $\mathbf{v}$ is the velocity field of the blood stream and $\mathbf{n}$ is the unit outward normal on Γ_{in}. On the outer boundary Γ_{out}, we have

$$\sigma \mathbf{n} + h \mathcal{K} \mathbf{u} = h \mathbf{f}, \tag{5}$$

where $\mathcal{K}$ is the tensor corresponding to the elastic response of the surrounding material so that $\mathcal{K}\mathbf{u} = k(\mathbf{u} \cdot \mathbf{n})\mathbf{n}$ for some given constant k and $\mathbf{f}$ is the force exerted on the pipe by external factors.

The flow of fluid inside the pipe is modelled by the Stoke's equation

$$\partial_t \mathbf{v} - \nu \Delta \mathbf{v} + \nabla p = \mathbf{g} \text{ and } \nabla \cdot \mathbf{v} = 0 \text{ in } \Omega, \tag{6}$$

where ν is the dynamic viscosity of fluid (blood) and $\mathbf{g}$ is the acceleration due to gravity.

2 Preliminaries and notations

2.1 Setting up a curvilinear coordinate system

We begin by choosing a suitable coordinate system that simplifies the computations even in the case of the most general geometry of the pipe. We assume a centre curve of the pipe to be known and given by an arc-length perameterized curve $\mathbf{c} \in \mathcal{C}^2([0, L], \mathbb{R}^3)$ for some positive real L. We may assume that $\mathbf{c}(0) = (0,0,0)^T$ and that $\mathbf{c}'(0) = (0,0,1)^T$. We denote the arc-length parameter by s.

We first need to build a right handed coordinate frame at each point $\mathbf{c}(s)$. It seems natural to take one of the coordinate directions to be $\mathbf{c}'(s)$ and the other two to be perpendicular to it. Let $\mathbf{e}_1$ be one of these vectors having unit length.

We could use the Frenet frame to define $\mathbf{e}_1(\theta, s) = \cos\theta \mathbf{N}(s) - \sin\theta \mathbf{B}(s)$ where $\mathbf{N}$ and $\mathbf{B}$ are the unit normal and the unit binormal of the curve $\mathbf{c}$, where we assumed that the curve has non vanishing curvature. Consider the surface $S(\theta, s) = \mathbf{c}(s) + r_\delta \mathbf{e}_1(\theta, s)$ for some $r_\delta > 0$. With the help of the Serret-Frenet formulas, one can see that even in this simple case, $(\partial S(\theta, s)/\partial\theta) \cdot (\partial S(\theta, s)/\partial s) \neq 0$ unless the curve is torsion free. In other words, the coordinate lines do not intersect at right angles when $\mathbf{c}$ has nonzero torsion.

In order to get coordinate lines on simple surfaces (as S considered above) to intersect perpendicularly, we put a requirement that the change in $\mathbf{e}_1$ as we travel along the central line, should be coplanar with $\mathbf{e}_1$ and $\mathbf{c}'(s)$. Then we have

$$\|\mathbf{e}_1\|^2 = 1 \Rightarrow \frac{\partial}{\partial s}\mathbf{e}_1 \cdot \mathbf{e}_1 = 0.$$

Also, as $\mathbf{c}'$ is perpendicular to $\mathbf{e}_1$ at each s, it follows that

$$\mathbf{e}_1 \cdot \mathbf{c}' = 0 \Rightarrow \frac{\partial}{\partial s}\mathbf{e}_1 \cdot \mathbf{c}' = -\mathbf{c}'' \cdot \mathbf{e}_1.$$

Finally, coplanarity condition gives

$$\frac{\partial}{\partial s}\mathbf{e}_1 \cdot (\mathbf{e}_1 \times \mathbf{c}') = 0 \Rightarrow \frac{\partial}{\partial s}\mathbf{e}_1 = (\frac{\partial}{\partial s}\mathbf{e}_1 \cdot \mathbf{c}')\mathbf{c}' + (\frac{\partial}{\partial s}\mathbf{e}_1 \cdot \mathbf{e}_1)\mathbf{e}_1 = -(\mathbf{c}'' \cdot \mathbf{e}_1)\mathbf{c}'.$$

We choose the initial value of $\mathbf{e}_1$ at $s = 0$ to be $(\cos\theta, \sin\theta, 0)^T$ for some $\theta \in [0, 2\pi]$. Then we can write the following partial differential equation system that defines $\mathbf{e}_1$

$$\frac{\partial}{\partial s}\mathbf{e}_1(\theta, s) = -(\mathbf{c}''(s) \cdot \mathbf{e}_1(\theta, s))\mathbf{c}'(s) \text{ and } \mathbf{e}_1(\theta, 0) = (\cos\theta, \sin\theta, 0)^T.$$

Defining $\mathbf{e}_2(\theta, s) = \mathbf{c}'(s) \times \mathbf{e}_1(\theta, s)$, the triple $\{\mathbf{e}_1(\theta, s), \mathbf{e}_2(\theta, s), \mathbf{c}'\}$ forms an orthonormal frame at each point $\mathbf{c}(s)$ for a given angle θ. As a result, we have

$$\begin{aligned}\frac{\partial}{\partial s}\mathbf{e}_2(\theta, s) &= \mathbf{c}''(s) \times \mathbf{e}_1(\theta, s) + \mathbf{c}'(s) \times \frac{\partial}{\partial s}\mathbf{e}_1(\theta, s) = \mathbf{c}''(s) \times (\mathbf{e}_2(\theta, s) \times \mathbf{c}'(s)) + 0 \\ &= \mathbf{c}''(s) \cdot \mathbf{c}'(s)\mathbf{e}_2(\theta, s) - \mathbf{c}''(s) \cdot \mathbf{e}_2(\theta, s)\mathbf{c}'(s) = -\mathbf{c}''(s) \cdot \mathbf{e}_2(\theta, s)\mathbf{c}'(s).\end{aligned}$$

The equations describing the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ suggest that the frame $\mathbf{c}', \mathbf{e}_1, \mathbf{e}_2$ is a so called 'rotation minimizing frame', see Bishop (1975); Klok (1986).

The initial condition reads

$$\mathbf{e}_2(\theta, 0) = \mathbf{c}'(0) \times \mathbf{e}_1(\theta, 0) = (-\sin\theta, \cos\theta, 0)^T.$$

We can assume a rotation matrix valued function R so that $\mathbf{e}_i(\theta, s) = R(s)\mathbf{e}_i(\theta, 0)$ for $i = 1, 2$. Then it is readily obtained that

$$\frac{\partial}{\partial \theta}\mathbf{e}_1(\theta, s) = \mathbf{e}_2(\theta, s) \text{ and } \frac{\partial}{\partial \theta}\mathbf{e}_2(\theta, s) = -\mathbf{e}_1(\theta, s).$$

The parameter θ corresponds to the orientation of the vectors $\mathbf{e}_1(\theta, s)$ and $\mathbf{e}_2(\theta, s)$ for fixed s with respect to some reference vector in the same disc perpendicular to the corresponding tangent vector $\mathbf{c}'(s)$ of the central curve. Note that in the torsion free case when $\mathbf{c}''$ is never zero and $\mathbf{c}''(0) = (1, 0, 0)^T$, the orthonormal frame is same as $\{\cos\theta\mathbf{N}(s) - \sin\theta\mathbf{B}(s), \sin\theta\mathbf{N}(s) + \cos\theta\mathbf{B}(s), \mathbf{c}'(s)\}$ where $\mathbf{N}$ and $\mathbf{B}$ are respectively the unit normal and the unit binormal of the curve $\mathbf{c}$.

We have two parameters, namely, θ and s, that describe the inner boundary of the shell of the pipe. Next we construct a coordinate system in the shell. In order to include the information of the layered structure of the pipe shell, we assume that the layers are given as level sets of a sufficiently smooth function $G : \mathbb{R}^3 \to \mathbb{R}$. The innermost layer is given as $\{\mathbf{x} \in \mathbb{R}^3 | G(\mathbf{x}) = 0\}$ while the outermost layer is given as $\{\mathbf{x} \in \mathbb{R}^3 | G(\mathbf{x}) = H\}$, where $H > 0$ is the mean thickness of the pipe shell. In particular, G could be assumed to have the form $G(\mathbf{x}) = d(\mathbf{x})a(\mathbf{x})$, where $d(\mathbf{x})$ gives the distance of $\mathbf{x}$ from Γ_{in} and $a(\mathbf{x})$ gives a suitable scaling so as to keep G constant over a given surface. The normal vector field across the layers is given by ∇G. Let another parameter n be such that it corresponds to the layer (we assume a continuum of layers filling up Γ) to which a given point belongs. In other words, let $n = G(\mathbf{x})$. Differentiating with respect to n, we get

$$1 = \nabla G(\mathbf{x}) \cdot \frac{\partial}{\partial n}\mathbf{x}.$$

On the other hand, the integral lines corresponding to the parameter n for fixed θ and s, have tangents $\partial\mathbf{x}/\partial n$ parallel to ∇G. Hence, we get the integral lines by solving ordinary differential equation

$$\frac{\partial}{\partial n}\mathbf{x}(n) = \frac{\nabla G(\mathbf{x}(n))}{|\nabla G(\mathbf{x}(n))|^2}.$$

The initial condition on such lines are that they originate from Γ_{in} where $n = 0$. In other words, for some $\theta \in [0, 2\pi]$ and $s \in [0, L]$,

$$\mathbf{x}(0) = \mathbf{c}(s) + r(\theta, s)\mathbf{e}_n(\theta, s)$$

where $r(\theta, s)$ is assumed to be a known radius function of the interior channel.

Thus, we have three parameters describing the pipe shell in a curvilinear coordinate system. The parameter n corresponds to the direction perpendicular to the layers, s corresponds to the tangential direction along the axial curve while θ corresponds to the direction tangential to the closed curve determined by fixed n and s.

The relation between the Cartesian and the curvilinear coordinate system in the wall is as given below.

$$\mathbf{x}(n,\theta,s) = \mathbf{c}(s) + r(\theta,s)\mathbf{e}_n(\theta,s) + \int_0^n \frac{\nabla G(\mathbf{x}(\tau))}{|\nabla G(\mathbf{x}(\tau))|^2}\mathrm{d}\tau.$$

2.2 Basis vectors and differential operators

In order to use the formulae mentioned in the previous section in terms of the new coordinates, we need to express vectors, tensors and differential operators in a suitable basis or cobasis. A detailed presentation of tensor algebra in curvilinear coordinates for application to continuum mechanics can be found in appendix D of Lurie (2005).

In what follows, we let $\partial_1 = \partial/\partial n$, $\partial_2 = \partial/\partial\theta$ and $\partial_3 = \partial/\partial s$. Also, we adopt Einstein's summation convention, that is, similar indices when appearing concurrently at both top and bottom positions in a term are assumed to be summed over the index set which is $\{1,2,3\}$ in our case.

We now define a set of main basis vectors for tangent vectors inside the shell structure. Let $\mathbf{x}_i = \partial_i\mathbf{x}$ for $i = 1,2,3$ and some $\mathbf{x} \in \Gamma$. This leads to the definition of the 3×3 metric tensor as $g_{ij} = \mathbf{x}_i \cdot \mathbf{x}_j$ for $i,j = 1,2,3$. Let g denote the matrix $[g_{ij}]$.

We may also define a set of cobasis vectors for the same space (cf. appendix D of Lurie (2005)) which are given as $\mathbf{x}^i = g^{ij}\mathbf{x}_j$, where g^{ij} is such that $g^{ij}g_{jk} = \delta^i_k$ with δ^i_k being the Kronecker delta. Note that basis vectors have bottom indices while cobasis vectors have top indices. In our case, both the vectors $\mathbf{x}_1$ and $\mathbf{x}^1$ are parallel to the normal direction across the layers at each point in the shell. This makes it easier for us to formulate the physical laws.

In order to express derivatives in a curvilinear system, we need the Christoffel symbols corresponding to the curvilinear system which are defined as $\Gamma^i_{jk} = \mathbf{x}^i \cdot \partial_j\mathbf{x}_k$ for $i,j,k = 1,2,3$. They are symmetric in the lower indices, i.e., $\Gamma^i_{jk} = \Gamma^i_{kj}$.

With the help of these relations, we can define the gradient operator as $\nabla = \mathbf{x}^i\partial_i$. We are now in a position to express quantities like gradient and divergence of tensors in our curvilinear coordinates. For any vector $\mathbf{v} = v_j\mathbf{x}^j = v^j\mathbf{x}_j$, its gradient is given as

$$\nabla\mathbf{v} = \left(\partial_i v_j - \Gamma^k_{ij}v_k\right)\mathbf{x}^i\mathbf{x}^j = \left(\partial_i v^j + \Gamma^j_{ik}v^k\right)\mathbf{x}^i\mathbf{x}_j.$$

Also, for any 2-tensor $\sigma = \sigma_{ij}\mathbf{x}^i\mathbf{x}^j = \sigma^{ij}\mathbf{x}_i\mathbf{x}_j$, its divergence is given as

$$\nabla\cdot\sigma = g^{ij}\left(\partial_i\sigma_{jk} - \Gamma^l_{ij}\sigma_{kl} - \Gamma^l_{ik}\sigma_{jl}\right)\mathbf{x}^k = (\partial_i\sigma^{ik} + \Gamma^i_{ij}\sigma^{jk} + \Gamma^k_{ij}\sigma^{ij})\mathbf{x}_k.$$

Similarly, the deformation (symmetric gradient) tensor for any vector is given as

$$2\,\mathrm{def}(\mathbf{v}) = (\nabla\mathbf{v} + (\nabla\mathbf{v})^*) = (\partial_i v_j + \partial_j v_i - 2\Gamma^k_{ij}v_k)\mathbf{x}^i\mathbf{x}^j.$$

2.3 Volume elements

The volume element with respect to the new variables is

$$\mathrm{d}v = \sqrt{det(g)}\mathrm{d}n\mathrm{d}\theta\mathrm{d}s. \tag{7}$$

Similarly, the surface element on a surface with fixed n is given as

$$\mathrm{d}S = \sqrt{g_{22}g_{33} - (g_{23})^2}\mathrm{d}\theta\mathrm{d}s. \tag{8}$$

2.4 Voight notation for tensors

As is evident from the above mentioned formulae, one has to deal with a good number of indices in each of the equations. The stress and strain tensors are each 3×3 tensors whereas the stiffness tensor is a $3\times 3\times 3\times 3$ tensor. This means it has 91 components. But, we notice that all these tensors have some symmetries which greatly reduce the number of independent components. We use the Voight notation to write only the independent quantities. In this notation, strain tensor $\varepsilon = \varepsilon_{ij}\mathbf{x}^i\mathbf{x}^j$ gets the representation

$$\boldsymbol{\varepsilon} = \left(\varepsilon_{11}, \sqrt{2}\varepsilon_{12}, \sqrt{2}\varepsilon_{13}, \varepsilon_{22}, \varepsilon_{33}, \sqrt{2}\varepsilon_{23}\right)^T.$$

We express the stress tensor $\sigma = \sigma^{ij}\mathbf{x}_i\mathbf{x}_j$ in the Voight notation by

$$\boldsymbol{\sigma} = \left(\sigma^{11}, \sqrt{2}\sigma^{12}, \sqrt{2}\sigma^{13}, \sigma^{22}, \sigma^{33}, \sqrt{2}\sigma^{23}\right)^T.$$

On the other hand, the symmetric gradient operator is represented as the matrix D defined as

$$D^T = \begin{bmatrix} \partial_1 - \Gamma^1_{11} & \frac{1}{\sqrt{2}}(\partial_2 - 2\Gamma^1_{12}) & \frac{1}{\sqrt{2}}(\partial_3 - 2\Gamma^1_{13}) & -\Gamma^1_{22} & -\Gamma^1_{33} & \frac{1}{\sqrt{2}}(-2\Gamma^1_{23}) \\ -\Gamma^2_{11} & \frac{1}{\sqrt{2}}(\partial_1 - 2\Gamma^2_{12}) & \frac{1}{\sqrt{2}}(-2\Gamma^2_{13}) & \partial_2 - \Gamma^2_{22} & -\Gamma^2_{33} & \frac{1}{\sqrt{2}}(\partial_3 - 2\Gamma^2_{23}) \\ -\Gamma^3_{11} & \frac{1}{\sqrt{2}}(-2\Gamma^3_{12}) & \frac{1}{\sqrt{2}}(\partial_1 - 2\Gamma^3_{13}) & -\Gamma^3_{22} & \partial_3 - \Gamma^3_{33} & \frac{1}{\sqrt{2}}(\partial_2 - 2\Gamma^3_{23}) \end{bmatrix}. \quad (9)$$

The divergence operator for a 2 tensor gets the matrix reprepresentation D^* which is the Hermitian conjugate of D with respect to surface measure defined in (8), i.e.

$$\int_\Gamma (D^*u)^T v\sqrt{g_{22}g_{33} - (g_{23})^2}\mathrm{d}\theta\mathrm{d}s = -\int_\Gamma u^T(Dv)\sqrt{g_{22}g_{33} - (g_{23})^2}\mathrm{d}\theta\mathrm{d}s$$

for all $u, v \in L^2(\mathbb{R}^2, \mathbb{R}^3)$ such that $u|_{s=0,L} = 0 = v|_{s=0,L}$. We have in this case,

$$D^* = \begin{bmatrix} \partial_1+\Gamma^i_{i1}+\Gamma^1_{11} & \frac{1}{\sqrt{2}}(\partial_2+\Gamma^i_{i2}+2\Gamma^1_{12}) & \frac{1}{\sqrt{2}}(\partial_3+\Gamma^i_{i3}+2\Gamma^1_{13}) & \Gamma^1_{22} & \Gamma^1_{33} & \frac{1}{\sqrt{2}}(2\Gamma^1_{23}) \\ \Gamma^2_{11} & \frac{1}{\sqrt{2}}(\partial_1+\Gamma^i_{i1}+2\Gamma^2_{12}) & \frac{1}{\sqrt{2}}(2\Gamma^1_{13}) & \partial_2+\Gamma^i_{i2}+\Gamma^2_{22} & \Gamma^2_{33} & \frac{1}{\sqrt{2}}(\partial_3+\Gamma^i_{i3}+2\Gamma^2_{23}) \\ \Gamma^3_{11} & \frac{1}{\sqrt{2}}(2\Gamma^3_{12}) & \frac{1}{\sqrt{2}}(\partial_1+\Gamma^i_{i1}+2\Gamma^1_{13}) & \Gamma^3_{22} & \partial_3+\Gamma^i_{i3}+\Gamma^3_{33} & \frac{1}{\sqrt{2}}(\partial_2+\Gamma^i_{i2}+2\Gamma^3_{23}) \end{bmatrix}$$

so that for $\sigma = \sigma^{ij}\mathbf{x}_i\mathbf{x}_j$, we have that $\nabla\cdot\sigma = D^*\left(\sigma^{11}, \sqrt{2}\sigma^{12}, \sqrt{2}\sigma^{13}, \sigma^{22}, \sigma^{33}, \sqrt{2}\sigma^{23}\right)^T$.

3 Modelling of elastic shell

In this section, we will obtain two-dimensional model of the elastic pipe shell, cf. Grotberg & Jensen (2004); Moireau et al. (2012). We follow the steps as in Kozlov & Nazarov (2016), i.e. we perform dimension reduction of the model given by equations (3), (4) and (5) by identifying a small parameter and assuming asymptotic expansions of the displacement vector, the stress and the strain tensors.

3.1 Asymptotic ansatz

A property of the pipe shell is that the thickness of the pipe wall is extremely small compared to some charateristic length l. So a natural choice for a small parameter in our case is $h =$ (mean thickness of wall)/ (characteristic length) $= H/l$. This also means that the physical quantities in question change much faster across the wall layers as compared to along the layers. This prompts us to introduce a fast variable $\xi = h^{-1}n \in [0, l]$.

We then assume that the displacement vector $\mathbf{u}$ admits the expansion

$$\mathbf{u}(n,\theta,s) = \mathbf{u}_0(\xi,\theta,s) + h\mathbf{u}_1(\xi,\theta,s) + h^2\mathbf{u}_2(\xi,\theta,s) + \cdots. \quad (10)$$

We denote the coordinate vector of $\mathbf{u}_k$ in the basis $(\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3)$ by $U_k = (u_{k1}, u_{k2}, u_{k3})^T$.

The differential operator D defined in (9) also gets the following expansion due to the variable change:

$$D = h^{-1}B\partial_\xi + E + hD_1 + h^2D_2 + \cdots, \tag{11}$$

where, $E = C + D_0$ and

$$B^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \end{bmatrix}, C^T = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}}\partial_2 & \frac{1}{\sqrt{2}}\partial_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \partial_2 & 0 & \frac{1}{\sqrt{2}}\partial_3 \\ 0 & 0 & 0 & 0 & \partial_3 & \frac{1}{\sqrt{2}}\partial_2 \end{bmatrix} \tag{12}$$

With $(\Gamma_{ij}^k)_l$ denoting the coefficient of h^l in the infinite series expression of Γ_{ij}^k, we have for $m \geq 0$

$$D_m^T = -\begin{bmatrix} (\Gamma_{11}^1)_m & \sqrt{2}(\Gamma_{12}^1)_m & \sqrt{2}(\Gamma_{13}^1)_m & (\Gamma_{22}^1)_m & (\Gamma_{33}^1)_m & \sqrt{2}(\Gamma_{23}^1)_m \\ (\Gamma_{11}^2)_m & \sqrt{2}(\Gamma_{12}^2)_m & \sqrt{2}(\Gamma_{13}^2)_m & (\Gamma_{22}^2)_m & (\Gamma_{33}^2)_m & \sqrt{2}(\Gamma_{23}^2)_m \\ (\Gamma_{11}^3)_m & \sqrt{2}(\Gamma_{12}^3)_m & \sqrt{2}(\Gamma_{13}^3)_m & (\Gamma_{22}^3)_m & (\Gamma_{33}^3)_m & \sqrt{2}(\Gamma_{23}^3)_m \end{bmatrix}. \tag{13}$$

We also use the following expansion:

$$\nabla G(\mathbf{x}(h\xi, \theta, s)) = \nabla G(\mathbf{x}(0, \theta, s)) + h\xi\frac{\partial}{\partial n}\nabla G(\mathbf{x}(0, \theta, s)) + \frac{h^2\xi^2}{2}\frac{\partial}{\partial n}\nabla G(\mathbf{x}(0, \theta, s)) + \cdots.$$

For the distance function d, we have that $\|\nabla d(\mathbf{x})\| = 1$ for all $\mathbf{x} \in \Gamma$. Also, $d(\mathbf{x}(0, \theta, s)) = 0$. Hence,

$$|\nabla G(\mathbf{x}(0, \theta, s))| = |a(\mathbf{x}(0, \theta, s))\nabla d(\mathbf{x}(0, \theta, s)) + d(\mathbf{x}(0, \theta, s))\nabla a(\mathbf{x}(0, \theta, s))| = |a(\mathbf{x}(0, \theta, s))|.$$

3.2 The two–dimensional model

Let F denote the coordinate vector of the hydrodynamic force $\mathbf{F}$ in the basis $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$. Let M be the leading term in the infinite series expression of g^{-1} with respect to h. So we have

$$M = \begin{bmatrix} |a(\mathbf{x}(0, \theta, s))|^2 & 0 & 0 \\ 0 & \frac{r_s^2 + (1 - r\mathbf{c}'' \cdot \mathbf{e}_n)^2}{(r_\theta^2 + r^2)(1 - r\mathbf{c}'' \cdot \mathbf{e}_n)^2 + r^2r_s^2} & \frac{-r_\theta r_s}{(r_\theta^2 + r^2)(1 - r\mathbf{c}'' \cdot \mathbf{e}_n)^2 + r^2r_s^2} \\ 0 & \frac{-r_\theta r_s}{(r_\theta^2 + r^2)(1 - r\mathbf{c}'' \cdot \mathbf{e}_n)^2 + r^2r_s^2} & \frac{r_\theta^2 + r^2}{(r_\theta^2 + r^2)(1 - r\mathbf{c}'' \cdot \mathbf{e}_n)^2 + r^2r_s^2} \end{bmatrix}.$$

Furthermore, assume $E^T = [E_1^T | E_2^T]$ with each block being a 3×3 matrix. So that,

$$E_2 = \begin{bmatrix} -(\Gamma_{22}^1)_0 & \partial_2 - (\Gamma_{22}^2)_0 & -(\Gamma_{22}^3)_0 \\ -(\Gamma_{33}^1)_0 & -(\Gamma_{33}^2)_0 & \partial_3 - (\Gamma_{33}^3)_0 \\ -\sqrt{2}(\Gamma_{23}^1)_0 & \frac{1}{\sqrt{2}}\partial_3 - \sqrt{2}(\Gamma_{23}^1)_0 & \frac{1}{\sqrt{2}}\partial_2 - \sqrt{2}(\Gamma_{23}^1)_0 \end{bmatrix}.$$

Similarly, we have

$$E_2^* = \begin{bmatrix} (\Gamma_{22}^1)_0 & (\Gamma_{33}^1)_0 & \frac{1}{\sqrt{2}}(2(\Gamma_{23}^1)_0) \\ \partial_2 + \Gamma_{i2}^i + (\Gamma_{22}^2)_0 & (\Gamma_{33}^2)_0 & \frac{1}{\sqrt{2}}(\partial_3 + (\Gamma_{i3}^i)_0 + 2(\Gamma_{23}^2)_0) \\ (\Gamma_{22}^3)_0 & \partial_3 + (\Gamma_{i3}^i)_0 + (\Gamma_{33}^3)_0 & \frac{1}{\sqrt{2}}(\partial_2 + (\Gamma_{i2}^i)_0 + 2(\Gamma_{23}^3)_0) \end{bmatrix}.$$

By $A = \begin{bmatrix} A_{\dagger\dagger} & A_{\dagger\ddagger} \\ A_{\dagger\ddagger}^T & A_{\ddagger\ddagger} \end{bmatrix}$, we denote the 6×6 matrix, with each block being 3×3 matrix, corresponding to the stiffness tensor such that

$$A\left(\varepsilon_{11}, \sqrt{2}\varepsilon_{12}, \sqrt{2}\varepsilon_{13}, \varepsilon_{22}, \varepsilon_{33}, \sqrt{2}\varepsilon_{23}\right)^T = \left(\sigma^{11}, \sqrt{2}\sigma^{12}, \sqrt{2}\sigma^{13}, \sigma^{22}, \sigma^{33}, \sqrt{2}\sigma^{23}\right)^T.$$

We set K to be the matrix representation of the tensor $\mathcal{K}$ in the appropriate basis. In our case, we have

$$K = \begin{bmatrix} k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{14}$$

With the above notations, we have the following theorem that gives us a two–dimensional model of the shell of a pipe.

Theorem 1. *In the formal asymptotic expansion*[1] *of the displacement vector* $\mathbf{u}$ *given in (10), we have that* U_0 *is independent of* ξ *and it satisfies the relation*

$$E_2^* Q E_2 U_0 - \bar{\rho} M \partial_t^2 U_0 - |\nabla G_0(\mathbf{x}(0,\theta,s))|^2 K U_0(\theta,s) = \rho_b F(\theta,s) - F_{ext}, \tag{15}$$

where $Q = \int\limits_0^l (A_{\ddagger\ddagger} - A_{\dagger\ddagger}^T A_{\dagger\dagger}^{-1} A_{\dagger\ddagger}) \mathrm{d}\xi$ *and* $\bar{\rho} = \int\limits_0^l \rho \mathrm{d}\xi$.

Proof. Proof can be found in Ghosh et al. (2018) and we present it here for readers convenience. Choosing the basis $(\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3)$ to express the vectors, equation (3) can be written as

$$D^* A D U = g^{-1} \rho \partial_t^2 U \text{ in } \Gamma, \tag{16}$$

where $U = [u_1, u_2, u_3]^T$ so that $\mathbf{u} = u_i \mathbf{x}^i$. The outer boundary condition is

$$B^T A D U = h(F_{ext} - g^{-1} K U) \text{ on } \Gamma_{\text{out}}, \tag{17}$$

where f denotes the column representing F_{ext} in the basis $(\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3)$.

After applying the substitution $\xi = h^{-1} n$ and the asymptotic ansatz (10) to (16) and (17), we compare the terms of various orders of h on both sides of the resulting equation. Comparing terms of order h^{-2} in (16) and h^{-1} in (17), we get the following system

$$B^T \partial_\xi A B \partial_\xi U_0 = (0,0,0)^T \text{ in } \Gamma,$$

$$B^T A B \partial_\xi U_0 = (0,0,0)^T \text{ on } \Gamma_{\text{out}}.$$

Solving this system and using the fact that $B^T A B$ is nonsingular, we obtain

$$\partial_\xi U_0 = (0,0,0)^T \text{ in } \Gamma.$$

This proves the first statement of the theorem that U_0 is independent of ξ.

Next we compare terms of order h^{-1} in (16) and h^0 in (17). Using the fact that $\partial_\xi U_0 = (0,0,0)^T$, we get

$$B^T \partial_\xi A(B \partial_\xi U_1 + E U_0) = (0,0,0)^T \text{ in } \Gamma,$$

$$B^T A(B \partial_\xi U_1 + E U_0) = (0,0,0)^T \text{ on } \Gamma_{\text{out}}.$$

Solving the above system, we get

$$B^T A(B \partial_\xi U_1 + E U_0) = (0,0,0)^T$$

$$\Leftrightarrow \partial_\xi U_1 = -(B^T A B)^{-1} B^T A E U_0. \tag{18}$$

Lastly, we compare terms of order h^0 in (16) and h in (17). This leads us to the following system

$$B^T \partial_\xi A(B \partial_\xi U_2 + E U_1 + D_1 U_0) + E^* A(B \partial_\xi U_1 + E U_0) = M \rho \partial_t^2 U_0 \text{ in } \Gamma, \tag{19}$$

[1] We assume (10) to be an asymptotic expansion which can be shown mathematically to be true but we leave this out as it is not relevant for the primary focus of this article.

$$B^T A(B\partial_\xi U_2 + EU_1 + D_1 U_0) = F_{ext} - |\nabla G_0|^2 K U_0 \text{ on } \Gamma_{\text{out}}. \tag{20}$$

On the other hand, the inner boundary conditions yield the following relation;

$$B^T A(B\partial_\xi U_2 + EU_1 + D_1 U_0) = \rho_b F \text{ on } \Gamma_{\text{in}}. \tag{21}$$

Note that Γ_{out} corresponds to $\xi = l = H/h$ while Γ_{in} corresponds to $\xi = 0$. Integrating (19) with respect to ξ from 0 to l and using (20) and (21), we have

$$F_{ext} - |\nabla G_0(\mathbf{x}(0,\theta,s))|^2 K U_0(\theta,s) - \rho_b F(\theta,s) + \int_0^l E^* A(B\partial_\xi U_1 + EU_0)\mathrm{d}\xi = M\bar{\rho}\partial_t^2 U_0(\theta,s).$$

Now (18) gives us that

$$E^* A(B\partial_\xi U_1 + EU_0) = E^*(A - AB(B^T AB)^{-1} B^T A)EU_0 = E_2^*(A_{\ddagger\ddagger} - A_{\dagger\ddagger}^T A_{\dagger\dagger}^{-1} A_{\dagger\ddagger})E_2 U_0.$$

Note that E^*, E and U_0 are independent of ξ. Hence we obtain

$$E_2^* Q E_2 U_0 - \bar{\rho} M \partial_t^2 U_0 - |\nabla G_0(\mathbf{x}(0,\theta,s))|^2 K U_0(\theta,s) = \rho_b F(\theta,s) - F_{ext}. \qquad \square$$

4 Examples of canonical shapes of pipes and their walls

In this section, we present a few simple cases of supplementary ones considered in Ghosh et al. (2018) and we look at the resulting expressions in the final model for each of these cases.

4.1 The straight cylinder

In this case, we assume the central curve to be a straight line. That is, $\mathbf{c}''(s) = 0$. We also have a fixed radius, so $r_\theta = 0 = r_s$. Then with the same initial conditions for the curve, we have for all $s \in [0, L]$ We get the orthonormal frame for all $s \in [0, L]$ and $\theta \in [0, 2\pi]$ as

$$\mathbf{e}_1(\theta,s) = \mathbf{e}_1(\theta,0) = (\cos\theta, \sin\theta, 0)^T \text{ and } \mathbf{e}_2(\theta,s) = \mathbf{e}_2(\theta,0) = (-\sin\theta, \cos\theta, 0)^T.$$

For the distance function d that measures distance from the innermost layer, we have

$$d(\mathbf{x}) = \sqrt{x_1^2 + x_2^2} - r \Rightarrow \nabla d(\mathbf{x}) = \mathbf{e}_1(\theta, 0),$$

where, $\mathbf{x} = (x_1, x_2, x_3)^T$ and θ is such that $\cos\theta = x_1/\sqrt{x_1^2 + x_2^2}$ and $\sin\theta = x_2/\sqrt{x_1^2 + x_2^2}$. Hence,

$$\nabla G(0,\theta,s) = a(\mathbf{x}(0,\theta,s))\mathbf{e}_1(\theta,0).$$

The matrices D_0 and M have the following expressions:

$$D_0^T = \begin{bmatrix} \frac{2\nabla a\cdot\mathbf{e}_1}{a^3} & \frac{\sqrt{2}r\nabla a\cdot\mathbf{e}_2}{a} & -\frac{\sqrt{2}\nabla a\cdot\mathbf{c}'}{a} & ar & 0 & 0 \\ -\frac{\nabla a\cdot\mathbf{e}_2}{ra^3} & -\frac{\sqrt{2}}{ra} & 0 & 0 & 0 & 0 \\ 5 - \frac{\nabla a\cdot\mathbf{c}'}{a^3} & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$M = \begin{bmatrix} |a|^2 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The differential operator matrices E_2 and E_2^* are given as

$$E_2 = \begin{bmatrix} ar & \partial_2 & 0 \\ 0 & 0 & \partial_3 \\ 0 & \frac{1}{\sqrt{2}}\partial_3 & \frac{1}{\sqrt{2}}\partial_2 \end{bmatrix} \text{ and } E_2^* = \begin{bmatrix} -ar & 0 & 0 \\ \partial_2 - \frac{r\nabla a\cdot\mathbf{e}_2}{a} & 0 & \frac{1}{\sqrt{2}}(\partial_3 + \frac{\nabla a\cdot\mathbf{c}'}{a}) \\ 0 & \partial_3 + \frac{\nabla a\cdot\mathbf{c}'}{a} & \frac{1}{\sqrt{2}}(\partial_2 - \frac{r\nabla a\cdot\mathbf{e}_2}{a}) \end{bmatrix}.$$

Note that all the functions in the above folmulae are evaluated at a point on Γ_{in}, where $\xi = 0$.

4.2 Pipe with curved axis and equally spaced layers

As in the first example, we take a fixed radius r for the pipe. Also, similar to the previous case, we assume equally spaced layers and hence $|G| = 1$. The matrices D_0 and M have the following expressions:

$$D_0^T = \begin{bmatrix} 0 & 0 & 0 & r & -c'' \cdot \mathbf{e}_1(1 - rc'' \cdot \mathbf{e}_1) & 0 \\ 0 & -\frac{\sqrt{2}}{r} & 0 & 0 & -\frac{c''\cdot\mathbf{e}_2}{r}(1 - rc'' \cdot \mathbf{e}_1) & 0 \\ 0 & 0 & \frac{\sqrt{2}c''\cdot\mathbf{e}_1}{1-rc''\cdot\mathbf{e}_1} & 0 & \frac{rc'''\cdot\mathbf{e}_1}{1-rc''\cdot\mathbf{e}_1} & \frac{\sqrt{2}rc''\cdot\mathbf{e}_2}{1-rc''\cdot\mathbf{e}_1} \end{bmatrix},$$

and

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{(1-rc''\cdot\mathbf{e}_1)} \end{bmatrix}.$$

The differential operator matrices E_2 and E_2^* are given as

$$E_2 = \begin{bmatrix} r & \partial_2 & 0 \\ -c'' \cdot \mathbf{e}_1(1 - rc'' \cdot \mathbf{e}_1) & -\frac{c''\cdot\mathbf{e}_2}{r}(1 - rc'' \cdot \mathbf{e}_1) & \partial_3 + \frac{rc'''\cdot\mathbf{e}_1}{1-rc''\cdot\mathbf{e}_1} \\ 0 & \frac{1}{\sqrt{2}}\partial_3 & \frac{1}{\sqrt{2}}\partial_2 + \frac{\sqrt{2}rc''\cdot\mathbf{e}_2}{1-rc''\cdot\mathbf{e}_1} \end{bmatrix}$$

and

$$E_2^* = \begin{bmatrix} -r & c'' \cdot \mathbf{e}_1(1 - rc'' \cdot \mathbf{e}_1) & 0 \\ \partial_2 - \frac{rc''\cdot\mathbf{e}_2}{1-rc''\cdot\mathbf{e}_1} & \frac{c''\cdot\mathbf{e}_2}{r}(1 - rc'' \cdot \mathbf{e}_1) & \frac{1}{\sqrt{2}}(\partial_3 - \frac{rc'''\cdot\mathbf{e}_1}{1-rc''\cdot\mathbf{e}_1}) \\ 0 & \partial_3 - \frac{2rc'''\cdot\mathbf{e}_1}{1-rc''\cdot\mathbf{e}_1} & \frac{1}{\sqrt{2}}(\partial_2 - \frac{3rc''\cdot\mathbf{e}_2}{1-rc''\cdot\mathbf{e}_1}) \end{bmatrix}.$$

4.3 Pipe with curved axis, variable radius and equally spaced layers

Here, we take a variable radius r for the pipe. Also, similar to the previous case, we assume equally spaced layers and hence $|G| = 1$. Normal to the boundary

$$\hat{n} = \nabla G = \frac{\beta\mathbf{e}_1 - r'c'}{\sqrt{\beta^2 + r'^2}}.$$

Relation between the Cartesian coordinates $\mathbf{x}$ and the curvilinear ones

$$\mathbf{x}(n, \theta, s) = c'(s) + r(s)\mathbf{e}_1(\theta, s) + \int_0^n \frac{\beta\mathbf{e}_1 - r'c'}{\sqrt{\beta^2 + r'^2}} d\tau.$$

We need basis vectors at $n = 0$. Let $\gamma = (\beta^2 + r'^2)^{-1/2}$, $\beta = 1 - rc''\mathbf{e}_1$, then the contravariant components are

$$x_1 = \gamma(\beta\mathbf{e}_1 - r'c'), \quad x_2 = r\mathbf{e}_2, \quad x_3 = \beta c' + r'\mathbf{e}_1,$$

and covariant components are

$$x^1 = \gamma(\beta\mathbf{e}_1 - r'c'), \quad x^2 = \frac{1}{r}\mathbf{e}_2, \quad x^3 = \frac{\beta c' + r'\mathbf{e}_1}{\beta^2 + r'^2}.$$

Note that if $r' = 0$ then $\gamma\beta = 1$. The matrice M has the following expression:

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & r^2 \end{bmatrix}.$$

The differential operator matrices E_2 and E_2^* are given as

$$\begin{bmatrix} r\gamma\beta & \partial_2 & \gamma^2 rr' \\ -r''\gamma\beta + \gamma r'(2r'c'' \cdot \mathbf{e}_1 + rc''' \cdot \mathbf{e}_1) - \gamma\beta^2 c'' \cdot \mathbf{e}_1 & -r^{-1}\beta c'' \cdot \mathbf{e}_2 & \partial_3 - \gamma^2((r'' + \beta c'' \cdot \mathbf{e}_1)r' - (2r'c'' \cdot \mathbf{e}_1 + rc''' \cdot \mathbf{e}_1)\beta) \\ \sqrt{2}rr'\gamma c'' \cdot \mathbf{e}_2 & \frac{1}{\sqrt{2}}\partial_3 - \sqrt{2}r^{-1}r' & \frac{1}{\sqrt{2}}\partial_2 + \sqrt{2}\gamma^2 rc'' \cdot \mathbf{e}_2\beta \end{bmatrix}$$

and

$$\begin{bmatrix} -r\gamma\beta & E_{12} & \sqrt{2}rr'\gamma c''\cdot\mathbf{e}_2 \\ \partial_2-\gamma^2 r\beta c''\cdot\mathbf{e}_2 & r^{-1}\beta c''\cdot\mathbf{e}_2 & E_{23} \\ -\gamma^2 rr' & E_{32} & \frac{1}{\sqrt{2}}\partial_2-\frac{3}{\sqrt{2}}\gamma^2 r\beta c''\cdot\mathbf{e}_2 \end{bmatrix},$$

respectively.

Here, $E_{12}=r''\gamma\beta-\gamma r'(2r'c''\cdot\mathbf{e}_1+rc'''\cdot\mathbf{e}_1)+r\beta^2c''\cdot\mathbf{e}_1$, $E_{23}=\frac{1}{\sqrt{2}}\partial_3+\frac{3}{\sqrt{2}}r^{-1}r'+\frac{1}{\sqrt{2}}r^2(r'(r''+\beta c''\cdot\mathbf{e}_1)-\beta(2r'c''\cdot\mathbf{e}_1+rc'''\cdot\mathbf{e}_1))$, $E_{32}=\partial_3+r^{-1}r'+2\gamma^2(r'(r''+\beta c''\cdot\mathbf{e}_1)-\beta(2r'c''\cdot\mathbf{e}_1+rc'''\cdot\mathbf{e}_1))$.

5 Analysis of the model

5.1 Stiffness tensor

Let us consider an elastic space weakened by the cylindrical void

$$\Omega=\{\mathbf{x}=(\mathbf{x}',x_3)\in\mathbb{R}^2\times\mathbb{R}:\tau=|\mathbf{x}'|=\sqrt{x_1^2+x_2^2}<R\}$$

of radius $R>0$. Assuming the transversal isotropy with the x_3-axis of a homogeneous stationary elastic material in $\Xi=\mathbb{R}^3\backslash\bar{\Omega}$, we write the equilibrium equations

$$-\nabla\cdot\sigma(\mathbf{u}^m;\mathbf{x})=\mathbf{0},\mathbf{x}\in\Xi, \tag{22}$$

where $\mathbf{u}^m=(\mathbf{u}'^m,u_3^m)$ is the three dimensional displacement vector and $\sigma(\mathbf{u})$ the corresponding stress tensor of rank 2 computed through the Hooke's law. In the Voigt-Mandel notation, the strain and stress

$$\varepsilon=(\varepsilon_{11},\varepsilon_{22},\sqrt{2}\varepsilon_{21},\sqrt{2}\varepsilon_{13},\sqrt{2}\varepsilon_{23},\varepsilon_{33})^T$$

and

$$\sigma^m=(\sigma_{11}^m,\sigma_{22}^m,\sqrt{2}\sigma_{21}^m,\sqrt{2}\sigma_{13}^m,\sqrt{2}\sigma_{23}^m,\sigma_{33}^m)^T$$

are related by $\sigma^m=A^m\varepsilon$, where

$$A^m=\begin{bmatrix} \lambda+2\mu & \lambda & 0 & 0 & 0 & \alpha \\ \lambda & \lambda+2\mu & 0 & 0 & 0 & \alpha \\ 0 & 0 & 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\beta & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\beta & 0 \\ \alpha & \alpha & 0 & 0 & 0 & \gamma \end{bmatrix}, \tag{23}$$

$\lambda\geq 0$ and $\mu>0$ are the classical Lamé constants in the $\mathbf{x}'$-plane while other elastic moduli $\alpha\geq 0$, $\beta>0$ and $\gamma>0$ are of no further use.

The particular problem on fluid (blood) flow requires to describe an interaction of the pipe wall with the surrounding material. In other words, we have to find out a relation between the pipe radial dilation

$$\mathbf{u}^m(\mathbf{x})=\mathbf{u}^w(\mathbf{x})=u_\tau^w\mathbf{e}_\tau,\mathbf{x}\in\partial\Omega, \tag{24}$$

and the traction

$$\sigma^m(\mathbf{u}^m;\mathbf{x})\mathbf{e}_\tau=\sigma^w(\mathbf{u}^w;\mathbf{x})\mathbf{e}_\tau,\mathbf{x}\in\partial\Omega, \tag{25}$$

where $\mathbf{e}_\tau=(\tau^{-1}\mathbf{x}',0)$ is the normal vector on $\partial\Omega$. Note that the equations (22) do not involve the inertia term $\gamma_m\partial_t^2u^m(x;t)$ because of the traditional and reasonable assumption on the Womersley number W_m to be small in comparison with the Womersley number W_w of the pipe wall. Moreover, owing to blood vessels are set in so-called vessel beds and in this way are enveloped by a loose cell material in order to prevent gyrations of the wall so that only the radial dilation is passed from the vessel wall to the muscle tissue we also assume the same behaviour for the pipe shell, cf. (24).

In general case, the mapping (25)$\mapsto$(24) is described with the help of the elasticity Neumann-to-Dirichlet operator which is rather complicated even in our canonical geometry. However, the above-accepted assumption on the low variability of all mechanical fields allows us to employ the asymptotic methods of singularly perturbed elliptic problems Maz'ya et al. (2000).

First of all, the transversal isotropy and the absence of the angular variable $\phi \in [0, 2\pi)$ in (24) prove that $u^m_\phi = \mathbf{u}^m \cdot \mathbf{e}_\phi = 0$ in Ξ_R. Furthermore, the low variability recognizes the variable z as a parameter and eliminates derivatives in z in the Cauchy formulas (2) as well as in equations (22) which take the form

$$-\frac{\partial}{\partial x_1}\sigma_{j1}(\mathbf{u}^m) - \frac{\partial}{\partial x_2}\sigma_{j2}(\mathbf{u}^m) = 0 \text{ in } \Xi_R, j = 1, 2, 3$$

or, in view of (23), become the plane elasticity system

$$-\mu\Delta_{\mathbf{x}'}u^m_j - (\lambda+\mu)\frac{\partial}{\partial x_j}\left(\frac{\partial u^m_1}{\partial x_1} + \frac{\partial u^m_2}{\partial x_2}\right) = 0, j = 1, 2, \tag{26}$$

$$-\beta\Delta_{\mathbf{x}'}u^m_3 = 0, \mathbf{x}' \in \mathbb{R}^2 \backslash \Omega. \tag{27}$$

At the same time, (24) reads componentwise as follows:

$$u^m_\tau(\mathbf{x}') = u^w_\tau(s), u^m_\phi(\mathbf{x}') = 0, \mathbf{x}' \in \Gamma_{\text{in}}, \tag{28}$$

$$u^m_3(\mathbf{x}') = 0, \mathbf{x}' \in \Gamma_{\text{in}}. \tag{29}$$

From (27) and (29) we derive that

$$u^m_3(\mathbf{x}') = 0, \mathbf{x}' \in \mathbb{R}^2 \backslash \Omega. \tag{30}$$

According to (28), the displacement field $\mathbf{u}^{m\prime}$ is axisymmetric and, therefore, in the polar coordinates (τ, ϕ), we have

$$u^m_\tau(\tau, \phi) = \frac{a}{\tau}, u^m_\phi(\tau, \phi) = 0, \tag{31}$$

$$\sigma^m_{\tau\tau}(u^{m\prime}; \tau, \phi) = -2\mu\frac{a}{\tau^2}, \sigma^m_{\phi\phi}(u^{m\prime}; \tau, \phi) = 2\mu\frac{a}{\tau^2}, \tag{32}$$

$$\sigma^m_{\tau\phi}(u^{m\prime}; \tau, \phi) = 0. \tag{33}$$

Finally, (28) gives $a = Ru^w_\tau(s)$ so that

$$\sigma^w(u^w; s, \phi)e_\tau = -\frac{2\mu}{R}u^w_\tau(s)e_\tau.$$

This relation gives the tensor $\mathcal{K}$ in (5) while

$$k = \frac{2\mu}{R}h^{-1} \text{ in } (14)$$

because $\mathcal{K}$ has the factor h in (5).

In this way, we need to assume that the elastic characteristics of the shell and the surrounding material are in the relation $h^{-2} : 1$.

5.2 Christoffel symbols and metric tensor calculated

General case. We present the values of several quantities used in our case. We use the big O notation to express the corresponding values in the new variable ξ. Here, we let $R = R(\mathbf{x}) = \nabla G(\mathbf{x})/|\nabla G(\mathbf{x})|^2$ and $\alpha = \alpha(\theta, s) = ((r_\theta(\theta,s)^2 + r(\theta,s)^2)(1 - r(\theta,s)\mathbf{c}''(s)\cdot\mathbf{e}_n(\theta,s))^2 + r(\theta,s)^2 r_s(\theta,s)^2)^{-1}$ in order to have relatively compact expressions. We write all the functions without their respective arguments in order to have tidy expressions. Subscript appearing in scalar functions, e.g. r_θ mean partial differentiation with respect to the subscripted variable. We first describe the main basis vectors.

$$\begin{aligned}
\mathbf{x}_1 &= R,\\
\mathbf{x}_2 &= r_\theta \mathbf{e}_n + r\mathbf{e}_\theta + \int_0^n R_\theta \mathrm{d}\tau = r_\theta \mathbf{e}_n + r\mathbf{e}_\theta + O(h),\\
\mathbf{x}_3 &= r_s \mathbf{e}_n + (1 - r\mathbf{c}''\cdot\mathbf{e}_n)\mathbf{c}' + \int_0^n R_s \mathrm{d}\tau = r_s \mathbf{e}_n + (1 - r\mathbf{c}''\cdot\mathbf{e}_n)\mathbf{c}' + O(h).
\end{aligned}$$

The metric tensor components are given next.

$$\begin{aligned}
g_{11} &= |R|^2,\\
g_{12} &= g_{21} = g_{13} = g_{31} = 0,\\
g_{22} &= |r_\theta \mathbf{e}_n + r\mathbf{e}_\theta + \int_0^n R_\theta \mathrm{d}\tau|^2 = r_\theta^2 + r^2 + O(h),\\
g_{33} &= |r_s \mathbf{e}_n + (1 - r\mathbf{c}''\cdot\mathbf{e}_n)\mathbf{c}' + \int_0^n R_s \mathrm{d}\tau|^2 = r_s^2 + (1 - r\mathbf{c}''\cdot\mathbf{e}_n)^2 + O(h),\\
g_{23} &= [r_\theta \mathbf{e}_n + r\mathbf{e}_\theta + \int_0^n R_\theta \mathrm{d}\tau]\cdot[r_s \mathbf{e}_n + (1 - r\mathbf{c}''\cdot\mathbf{e}_n)\mathbf{c}' + \int_0^n R_s \mathrm{d}\tau] = r_\theta r_s + O(h).
\end{aligned}$$

6 Acknowledgement

The work is supported by Linköping University, and by RFBR grant 16-31-60112.

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